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Complete Integrability of Coupled KdV and NLS Equations

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Painlevé analysis is performed for the coupled system of nonlinear partial differential equations consisting of the KdV equation and NLS equation initially studied by Nishikawa. Various possibilities for the constants occurring in the system are explored, paying attention to the integrability property. This equation occurring in the field of plasma physics satisfies all the requirements of Painlevé analysis and can be ascertained to be completely integrable, though no Lax pair is known for it.

1. INTRODUCTION

One of the most frequently occurring equations in plasma physics is the coupled system of nonlinear Schrödinger equation and KdV, considered by Nishikawa (1974). Lax pair is not known for the system. So one does not have any idea about the complete integrability of this system. On the other hand, recently Conte (1988) suggested that it may be more fruitful to make a Painlevé analysis for such a system, because there is no other avenue for analyzing such a system. In a recent communication some important results have been obtained for the coupled system of Boussinesq and nonlinear Schrödinger equations (Roy Chowdhury and Chanda, 1987), which also does not possess a Lax pair. In this paper we analyze the equations of Nishikawa from the viewpoint of Painlevé analysis following Weiss (1984).

2. FORMULATION

The equations are written as

$$\begin{aligned} i\psi_t + \psi_{xx} + u\psi &= 0 \\ -i\chi_t + \chi_{xx} + u\chi &= 0 \\ u_t + u_x + u_{xxx} + \beta_{uu}x &= (\psi\chi)_x \end{aligned} \tag{1}$$

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The Painlevé analysis proceeds by assuming a Laurent expansion of the nonlinear field variables over the solution manifold, $\phi(x, t)=0$. So we set

$$\begin{aligned} u &= \sum_{j=0}^{\infty} u_j \phi^{p+j} \\ \psi &= \sum_{j=0}^{\infty} \psi_j \phi^{q+j} \\ \chi &= \sum_{j=0}^{\infty} \chi_j \phi^{r+j} \end{aligned} \tag{2}$$

Matching the leading terms in each of equations (1), we get

$$p=q=r=-2$$

whereas the coefficients of these terms yield

$$\begin{aligned} u_0 &= -6\phi_x^2 \\ \psi_0\chi_0 &= 18\phi_x^4(\beta-2\alpha) \end{aligned} \tag{3}$$

From the terms next to leading order we get the following system matrix T, whose determinant upon vanishing will yield an equation for the resonance positions:

$$T=\begin{vmatrix} (m-2)(m-3)\phi_x^2+u_0 & \psi_0 & 0 \\ 0 & \chi_0 & (m-2)(m-3)\phi_x^2+u_0 \\ -(m-4)\phi_x\chi_0 & \alpha(m-2)(m-3)(m-4)\phi_x^3+\beta u_0(m-4)\phi_x & -\psi_0(m-4)\phi_x \end{vmatrix} \tag{4}$$

Using equation (3), we find that det $T=0$ leads to the equation

$$m(m+1)(m-4)(m-5)(m-6)[6\beta-\alpha(m^2-5m+12)]=0 \tag{5}$$

So we get resonance at

$$m=0, -1, 4, 5, 6, \text{ and } m_1, m_2 \tag{6}$$

where m_1, m_2 are

$$m_{1,2}=\frac{5\alpha \pm (24\alpha\beta-23\alpha^2)^{1/2}}{2\alpha} \tag{7}$$

Equation (3) already gives an indication that one of ψ_0 or χ_0 is arbitrary corresponding to $m=0$. First we consider the simplest values of α, β which will make $m_{1,2}$ integral. Such a situation arises when $\alpha=\beta$, and $m_{1,2}=3, 2,$

$$u_0=-6\phi_x^2, \qquad \psi_0\chi_0=-18\alpha\phi_x^4 \tag{8}$$

On the other hand, for $m=1$,

$$\begin{aligned} u_1 &= -30\phi_{xx} \\ \psi_1 &= -\frac{8\phi_{xx}\psi_0}{\phi_x^2}-\frac{\psi_{0x}}{\phi_x}-\frac{i}{2}\frac{\psi_0}{\phi_x^2}\phi_t \\ \chi_1 &= -\frac{8\phi_{xx}\chi_0}{\phi_x^2}-\frac{\chi_{0x}}{\phi_x}+\frac{i}{2}\frac{\chi_0\phi_t}{\phi_x^2} \end{aligned} \tag{9a}$$

Proceeding with the situations $m=2$, 3, etc., we obtain the coefficients recursively. For this we write down the general recursion relation among the various coefficients obtained by equating similar power of ϕ. These are

$$\begin{aligned} & i[\psi_{m-2,t}+\psi_{m-1}\phi_t(m-3)]+\psi_{m-2,xx}+2(m-3) \\ &\quad \times\psi_{m-1,x}\phi_x+\psi_{m-1}(m-3)\phi_{xx}+(m-2)(m-3)\times\psi_m\phi_x^2 \\ &\quad +\sum_k U_{m-k}\psi_k=0 \end{aligned} \tag{9b}$$

$$\begin{aligned} & -i\chi_{m-2,t}-\chi_{m-1}\phi_t(m-3)+\chi_{xx}+(m-2)(m-3)\chi_m\phi_x^2 \\ &\quad +\psi_k U_{m-k}\phi_k=0 \end{aligned} \tag{9c}$$

$$\begin{aligned} & U_{m-3,t}+U_{m-2}\phi_t(m-4)+U_{m-3x}+U_{m-2}\phi_x x(m-4) \\ &\quad +\alpha[U_{m-3,xxx}+3U_{m-2,xx}(m-4)\phi_x+3U_{m-2,x}(m-4)\phi_{xx} \\ &\quad +3U_{m-1,x}(m-3)(m-4)\phi_x^2+U_{m-2}(m-4)\phi_{xxx}+3U_{m-1}(m-3) \\ &\quad \times(m-4)\phi_x\phi_{xx}+U_m(m-2)(m-3)(m-4)\phi_x^3] \\ &\quad +\beta/2\sum_k[(U_{m-k-1}U_k)_x+U_{m-k}\times U_k(m-4)\phi_x] \\ & =\sum_k(\psi_{m-1-k}\psi_k)_x+\sum_k\psi_{m-k}\chi_k(m-4)\phi_x \end{aligned} \tag{9d}$$

Then, for $m=2$,

$$\begin{aligned} U_2\psi_0+\chi_2U_0 &= H \\ U_2\chi_0+\chi_2U_0 &= L \\ -U_2(2\phi_xU_0)+(2\phi_x\chi_0)\psi_2+(2\phi_x\psi_0)\chi_2 &= M \end{aligned} \tag{10}$$

where

$$
\begin{aligned}
H &= -i(\psi_{0t} - \psi_1\phi_t) - \psi_{0xx} + 2\psi_{1x}\phi_x + \psi_1\phi_{xx} - U_1\psi_1 \\
L &= i(\chi_{0t} - \chi_1\phi_t) - \chi_{0xx} + 2\chi_{1x}\phi_x + \chi_1\phi_{xx} - U_1\chi_1 \\
M &= 2u_0\phi_t + 2\phi_x u_0 + 6\phi_x u_{0xx} + 6u_{0x}\phi_{xx} - 6u_{1x}\phi_x^2 \\
&\quad + 2u_0\phi_{xxx} - 6u_1\phi_x\phi_{xx} - (u_0u_1)_x + \phi_x u_1^2 + (\chi_1\psi_0)_x \\
&\quad + (\psi_1\chi_0)_x - 2\phi_x\psi_1\chi_1
\end{aligned}
\tag{11}
$$

In equation (10) the determinant of u_2, ψ_2, χ_2 on the lhs vanishes. So all of them are not determined.

$m = 3$;

$$
\begin{aligned}
i\psi_{1t} + \psi_{1xx} + u_2\psi_1 + u_1\psi_2 &= -(u_3\psi_0 + u_0\psi_3) \\
-i\chi_{1t} + \chi_{1xx} + u_2\chi_1 + u_1\chi_2 &= (u_3\chi_0 + u_0\chi_3) \\
u_{0t} - u_1\phi_t + u_{0x} - \phi x_1^u + [u_{0xxx} - 3\phi_x u_{1xx}\phi_x - 3u_{1x}\phi_{xx} - u_1\phi_{xxx}] \\
+ \tfrac{1}{2}[2(u_2u_0)_x + 2u_1u_{1x} - 2\phi x^u 3_0^u - 2\phi_x u_2 u_1] \\
= (\psi_2\chi_0 + \chi_1\psi_1 + \psi_0\chi_2)_x - \phi_x(\psi_3\chi_0 + \psi_2\chi_1 + \psi_1\chi_2 + \psi_0\chi_3)
\end{aligned}
\tag{12}
$$

Again one can check that the same situation as $m = 2$ is present and one cannot determine all the coefficients (u_3, ψ_3, χ_3). It then becomes a simple routine exercise to proceed up to $m = 6$, and verify the arbitrariness of the required number of coefficients. So we can ascertain that at least for $\beta = \gamma$, we can satisfy the Cauchy–Kowalevskya theorem and say that system (1) is completely integrable. Let us now consider another situation for which $\beta = 2\gamma$. In this case m_1, $m_2 = 0$, 5. The resonance positions at 0, 5 are repeated twice, while others remain the same and the number of arbitrary functions are thereby reduced. Equation (3) indicates that only one of the coefficients (u_0, ψ_0, χ_0) is arbitrary. But a double resonance at $m = 0$ requires the arbitrariness of at least two of these, which is not the case. A similar situation takes place at $m = 5$. So in the case $\beta = 2$ we can say that equation (1) has a special Painlevé property.

3. DISCUSSION

In the above we have presented a Painlevé analysis for a coupled set of nonlinear equation whose Lax pair is still not known. Surprisingly, the set of equations conform to all the criterion of Cauchy–Kowalevskya. Furthermore very recently it has been stressed that Painlevé analysis should be performed not only for integrable systems, but also for non-integrable ones. Lastly, an algebrogeometric technique has been developed by the Russian School to integrate (i.e., to find periodic and multi-solitons) the system of equation under consideration.

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